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A. D. Young, B.A. and N. E. Winterbottom, B.Sc.

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High-speed Flow in Smooth Cylindrical Pipes of Circular Section

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A. D. YOUNG, B.A. and N. E. WINTERBOTTOM, B.Sc.

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Summary.—A theory is developed for investigating the flow in smooth circular pipes at high speeds, taking into account the development of the boundary layer which is assumed to be turbulent from the entry. The calculations give the length of the pipe for which " choking " occurs at the outlet, and the distributions of velocity, pressure, density and skin friction along the pipe. The calculations have been made to cover entry Mach numbers ranging from 0·6 to 2·5 and entry Reynolds numbers from 10^4 to 10^8. An approximate analysis by Hawthorne (1942)[1], in which the development of the boundary layer is neglected, gives results which do not differ seriously from the results of these calculations, but the application of Hawthorne's results requires an accurate estimate of the mean value of the local frictional coefficient to be made. Comparison with some experimental results due to Frössel (1936)[5] shows satisfactory agreement.

1. *Introduction.*—In Ref. 1, Hawthorne has presented the results of calculations of the flow in straight pipes at high speeds based on a simplified theory in which the development of the boundary layer is neglected, pressure, density and velocity are assumed constant across any section of the pipe and the local frictional coefficient is assumed constant along the length of the pipe. The results of such calculations are of considerable theoretical and practical interest, hence it was felt desirable to determine the order of error involved in the above approximations. Calculations have therefore been made of the flow in smooth straight cylindrical pipes of circular cross-section, taking account of the development of the boundary layer, the variation of density across any section of the pipe and the variation in the frictional coefficient along the pipe. The theory upon which these calculations are based is a development of that adopted for the calculation of the drag of aerofoils in compressible flow[2]. The calculations have been made to cover values of the entry Reynolds number (*see* § 2 for definition) of 10^4, 10^6 and 10^8, and values of the entry Mach number ranging from 0·6 to 2·5. The boundary layer has been assumed to be completely turbulent in every case. The main item of interest in the results of these calculations is, as in Hawthorne's calculations, the " choking " length, or in the case of supersonic entry conditions, the length of pipe from the entry to the compression shock ; by-products are, however, the velocity, density, pressure and friction distributions along the pipe.

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2. *Notation*

r radius of the pipe cross-section,

x distance along the pipe axis from the entry,

y distance measured inwards normal to the pipe surface,

δ boundary layer thickness,

u, ρ velocity parallel to the axis, and density, respectively, in the boundary layer,

$U, \mathscr{R}$ velocity and density, respectively, outside the boundary layer or along the pipe axis,

p pressure in the pipe, assumed constant across a section,

τ frictional intensity at the pipe surface,

θ momentum thickness of the boundary layer $= \int_0^\delta \frac{\rho u}{\mathscr{R} U}\left(1 - \frac{u}{U}\right)\left(1 - \frac{y}{r}\right) dy$,

δ^* displacement thickness of the boundary layer $= \int_0^\delta \left(1 - \frac{\rho u}{\mathscr{R} U}\right)\left(1 - \frac{y}{r}\right) dy$,

$H =$ δ^*/θ,

$\zeta =$ $\sqrt{\frac{\mathscr{R} U^2}{\tau}}$,

Q mass flow per sec. along the pipe,

γ ratio of the specific heat at constant pressure to the specific heat at constant volume $= 1{\cdot}405$,

M Mach number outside the boundary layer or at the centre of the pipe $= U \Big/ \sqrt{\frac{\gamma p}{\mathscr{R}}}$,

R_e entry Reynolds number $= \frac{Q}{\pi r \mu_w} = \frac{U_0 r \mathscr{R}_0}{\mu_w}$,

μ_w coefficient of viscosity at the wall, assumed constant along the pipe,

suffix 0 denotes quantities at the pipe entry,

suffix c denotes quantities at the point where choking or shock occurs,

suffix p denotes quantities at the point where fully established pipe flow begins,

c_f skin friction coefficient $= 2\tau/\mathscr{R}_0 U_0^2$

f local frictional coefficient $= 2\tau/\mathscr{R} U^2$,

$\bar{f}$ mean local frictional coefficient, averaged over the choking length $= \frac{1}{x_c}\int_0^{x_c} \frac{2\tau}{\mathscr{R} U^2} dx$.

3. *Theory.*—3.1. *Theory for Cases where the Boundary Layer does not Extend to the Centre of the Pipe.*—The momentum equation of the boundary layer in a straight cylindrical pipe of circular cross-section is

$$\frac{d}{dx}\int_0^\delta 2\pi\,(r-y)\,\rho u^2 dy - U\int_0^\delta \frac{d}{dx}\Big[\,2\pi\,(r-y)\,\rho y\,\Big]\,dy$$

$$+\frac{d}{dx}\int_0^\delta p\cdot 2\pi\,(r-y)\,dy - p_\delta\frac{d}{dx}\int_0^\delta 2\pi\,(r-y)\,dy + \tau 2\pi r = 0. \quad .. \quad .. \quad (1)$$

The usual assumption in boundary layer theory is now made, namely that p is constant across the layer and therefore across the pipe.

Writing $\mathscr{R}$ for the density outside the boundary layer, we have

$$\frac{\partial p}{\partial x} = -\mathscr{R}U.U', \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2)$$

where the dash denotes differentiation with respect to x.

Equation (1) then becomes

$$-\frac{d}{dx}\int_0^\delta 2\pi\,(r-y)\,\rho u\,(U-u)\,dy + \frac{d}{dx}\int_0^\delta 2\pi\,(r-y)\,\rho uU dy$$

$$-U\int_0^\delta \frac{d}{dx}\Big[\,2\pi\,(r-y)\,\rho u\,\Big]\,dy - \mathscr{R}U.U'\int_0^\delta 2\pi\,(r-y)\,dy + \tau.2\pi r = 0,$$

or

$$-\frac{d}{dx}\int_0^\delta 2\pi\,(r-y)\,\rho u\,(U-u)\,dy + U'\int_0^\delta 2\pi\,(r-y)\,(\rho u - \mathscr{R}U)\,dy + \tau.2\pi r = 0,$$

and hence

$$\frac{d}{dx}\left\{\mathscr{R}U^2\int_0^\delta 2\pi\,(r-y)\,\frac{\rho u}{\mathscr{R}U}\Big(1-\frac{u}{U}\Big).dy\right\}$$

$$+\frac{U'}{U}.\mathscr{R}U^2\int_0^\delta 2\pi\,(r-y)\Big(1-\frac{\rho u}{\mathscr{R}U}\Big)\,dy = \tau.2\pi r. \quad .. \quad .. \quad .. \quad .. \quad (3)$$

This may be written

$$\frac{d}{dx}\Big[\mathscr{R}U^2\,2\pi r\theta\Big] + \frac{U'}{U}\Big[\mathscr{R}U^2.2\pi r\,\delta^*\Big] = \tau 2\pi r,$$

where θ is the momentum thickness of the boundary layer

$$= \int_0^\delta \frac{\rho u}{\mathscr{R}U}\Big(1-\frac{u}{U}\Big)\Big(1-\frac{y}{r}\Big).dy,$$

and δ^* is the displacement thickness of the boundary layer

$$= \int_0^\delta \Big(1-\frac{\rho u}{\mathscr{R}U}\Big)\Big(1-\frac{y}{r}\Big).dy.$$

Writing $H = \delta^*/\theta$, the momentum equation finally reduces to

$$\theta' + \theta\left[\frac{\mathscr{R}'}{\mathscr{R}} + (H+2)\,\frac{U'}{U}\right] = \frac{1}{\zeta^2}, \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (4)$$

where $\zeta^2 = \mathscr{R}U^2/\tau$.

Continuity of mass flow along the pipe requires that

$$\int_0^r 2\pi\rho u\,(r-y).\,dy = \text{const.} = \mathscr{R}_0 U_0 \pi r^2 = Q, \text{ say}, \qquad (5)$$

where suffix 0 refers to conditions at the inlet, and Q is the mass flow per second in the pipe.

Equation (5) may be written

$$\mathscr{R}U = \frac{Q}{\pi r^2 - 2\pi r H\theta}. \qquad (6)$$

Bernouilli's equation for compressible flow gives us

$$\frac{\gamma}{\gamma-1}\frac{p}{\mathscr{R}} + \tfrac{1}{2}U^2 = \frac{\gamma}{\gamma-1}\cdot\frac{p_0}{\mathscr{R}_0} + \tfrac{1}{2}U_0^2 \qquad (7)$$

and, as long as fully developed pipe flow is not attained (i.e., as long as the boundary layer does not extend to the centre of the pipe), the adiabatic relation between p and $\mathscr{R}$ holds, viz.

$$\frac{p}{\mathscr{R}^\gamma} = \frac{p_0}{\mathscr{R}_0{}^\gamma}. \qquad (8)$$

We now make all quantities non-dimensional by expressing lengths in terms of r, and U, p, $\mathscr{R}$, etc. in terms of U_0, p_0, $\mathscr{R}_0$, etc. Equation (4) then remains unchanged in form, viz.

$$\theta' + \theta\left[\frac{\mathscr{R}'}{\mathscr{R}} + (H+2)\frac{U'}{U}\right] = \frac{1}{\zeta^2}. \qquad (4a)$$

Equation (6) becomes

$$\mathscr{R}U = \frac{1}{1-2H\theta}. \qquad (6a)$$

Equation (7) becomes

$$\frac{p}{\mathscr{R}} + \frac{\gamma-1}{2}U^2M_0^2 = 1 + \frac{\gamma-1}{2}M_0^2, \qquad (7a)$$

where M_0 is the entry Mach number $= U_0/a_0$, and a_0 is the speed of sound at the entry.

Equation (8) becomes

$$\frac{p}{\mathscr{R}^\gamma} = 1. \qquad (8a)$$

From equations (7a) and (8a) we have

$$\mathscr{R} = \left[1 + \frac{\gamma-1}{2}M_0^2\,(1-U^2)\right]^{1/(\gamma-1)} \qquad (9)$$

and

$$U\mathscr{R} = U\left[1 + \frac{\gamma-1}{2}M_0^2\,(1-U^2)\right]^{1/(\gamma-1)}. \qquad (10)$$

We now assume that H is a constant $= 1{\cdot}4$ as for incompressible flow (*cf.* Squire & Young, R. & M. 1938[3]) and we obtain from equation (6a)

$$\theta' = \frac{U\mathscr{R}' + U'\mathscr{R}}{2H(U\mathscr{R})^2}, \qquad (11)$$

and from equation (9) we have

$$\frac{\mathscr{R}'}{\mathscr{R}} = \frac{-M_0^2U^2}{\left[1 + \frac{\gamma-1}{2}M_0^2\,(1-U^2)\right]}\cdot\frac{U'}{U}. \qquad (12)$$

Substituting for θ' and $\mathscr{R}'/\mathscr{R}$ from equations (11) and (12), equation (4a) can be finally reduced to

$$\frac{U'}{U} G(U, M_0) = \frac{1}{\zeta^2} , \quad \ldots \quad (13)$$

where

$$G(U, M_0) = \frac{H+2}{2H} - \frac{H+1}{2HU\mathscr{R}} - \frac{M_0^2 U^2}{2H\left[1 + \frac{\gamma-1}{2} M_0^2 (1-U^2)\right]}$$

$$= 1{\cdot}214 - \frac{0{\cdot}858}{U\mathscr{R}} - \frac{M_0^2 U^2}{2{\cdot}8\,[\,1 + 0{\cdot}2025\, M_0^2 (1-U^2)\,]} , \quad \ldots \quad (14)$$

putting $H = 1{\cdot}4$, and $\gamma = 1{\cdot}405$.

$U\mathscr{R}$ can be determined as a function of U and M_0 from equation (10) and hence the function $G(U, M_0)$ can be determined.

To solve equation (13) a relation between ζ and U is required. This is obtained by using the assumption made by Young and Winterbottom[2] namely, that

$$\frac{U\mathscr{R}\theta}{\mu_w} = A\,(e^{B\zeta} - 1),$$

where $A = 0{\cdot}2454$ and $B = 0{\cdot}3914$,

μ_w is the viscosity coefficient at the wall which may be taken as constant (*see* Ref. 2).

Written non-dimensionally, this expression becomes

$$U\mathscr{R}\theta = \frac{A\,(e^{B\zeta} - 1)}{R_e} , \quad \ldots \quad (15)$$

where R_e is the entry Reynolds number defined by

$$R_e = \frac{Q}{\pi r \mu_w} , \quad \ldots \quad (16)$$

From (15), (6a) and (10) we have

$$1 + \frac{2HA\,(e^{B\zeta} - 1)}{R_e} = U\mathscr{R} = U\left[1 + \frac{\gamma-1}{2} M_0^2 (1-U^2)\right]^{1/(\gamma-1)} \quad \ldots \quad (17)$$

and hence we can determine ζ as a function of U, M_0 and R_e.

By means of suitably tabulated or plotted values of the functions of $G(U, M_0)$ and ζ for the required range of values of M_0 and R_e, equation (13) can be solved for U by a step-by-step integration. From the distribution of U the distributions of $\mathscr{R}$, p, θ and ζ can then be derived from equations (9), (8a), (6a) and (7).

From equation (10) it is easily seen that $U\mathscr{R}$ is a maximum when

$$U^2 = \frac{2 + (\gamma-1) M_0^2}{(\gamma+1) M_0^2} , \quad \ldots \quad (18)$$

But the local Mach number corresponding to a velocity U is given, from equation (7a), by

$$M^2 = \frac{2U^2 M_0^2}{2 + (\gamma-1)(1-U^2) M_0^2} \quad \ldots \quad (19)$$

and equation (18) therefore expresses the condition that the local Mach number in the centre of the pipe is unity, i.e. U there is equal to the local speed of sound.

Examining the expression for $G(U, M_0)$ given in equation (14) we see that

$$2H\,G(U, M_0) = H + 2 - \frac{H+1}{U\mathscr{R}} - M^2 \quad \ldots \quad (20)$$

From (6a) we see, since $\theta > 0$, that $U\mathscr{R} > 1$; it follows that when $M = 1$

$$2H\,G(U, M_0) = (H+1)\left[1 - \frac{1}{U\mathscr{R}}\right] > 0 .$$

At the entry, however, where $M = M_0$, and $U\mathscr{R} = 1$,

$$\begin{aligned} 2H\,G(U, M_0) &= 1 - M_0^2 \\ &> 0, \text{ if } M_0 < 1 \\ &< 0, \text{ if } M_0 > 1. \end{aligned}$$

It follows that $G(U, M_0) = 0$, for some value of $M > 1$, for both $M_0 < 1$ and $M_0 > 1$. From (13) it will be seen that U' becomes infinite when $G(U, M_0) = 0$, and the smooth continuous development of the flow is indicated by the theory, therefore, to break down at this point. The only approximations involved in the theory are that H is constant and the pressure is constant across a cross-section of the pipe, neither of these approximations are liable to lead to serious differences from the actual flow at any stage whilst the flow is developing smoothly, it can therefore be inferred that the discontinuity indicated by the theory must correspond to a fundamental change in the flow characteristics. It follows from the discussion above, that if the entry flow is subsonic, the velocity of the flow outside the boundary layer exceeds the local speed of sound before this stage is reached, but, if the entry flow is supersonic, the breakdown is reached before the velocity is reduced to the speed of sound. In effect, the breakdown occurs when some mean Mach number of the flow across the whole cross-section of the pipe is equal to unity. From equations (20) and (6a) it will be seen that this mean Mach number* is given by

$$M_{mean} = [\,M^2 - (H+1)\,2H\theta\,]^{1/2} . \quad \ldots \quad (21)$$

A discussion in physical terms of the significance of this breakdown in the smooth development of the flow will be given later (§ 5), the distance from entry to a point where this occurs will be referred to hereafter as the " choking " distance.

It will be seen, since U' is infinite at the " choking " point, that the step-by-step integration of equation (13) becomes increasingly difficult as that point is approached. At a convenient point nearby the step-by-step integration was therefore stopped and the solution was completed as follows. When plotting $G(U, M_0)$ as a function of U it was found in every case to be a nearly linear function of U in the region of the limiting value of U (denoted by U_c). Therefore, writing

$$G(U, M_0) = \lambda(U_c - U),$$

equation (13) becomes

$$\lambda U' \left(\frac{U_c}{U} - 1\right) = \frac{1}{\zeta^2} . \quad \ldots \quad (22)$$

* A discussion of the value of the arithmetic mean square Mach number at the critical stage is given in the Appendix.

The value of ζ changes very little near the limiting value of U, equation (22) was therefore integrated on the assumption that $\zeta = \text{const.} = \zeta_I$; the suffix I refers to the point at which the step-by-step process had been stopped. The solution is easily seen to give

$$\frac{x_c - x_I}{\zeta_1^2} = \lambda \left[U_c \log \frac{U_c}{U_I} - (U_c - U_I) \right] .$$

$$= \frac{\lambda}{U_c} (U_c - U_I)^2, \text{ approx.} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (23)$$

This equation therefore gives the additional length from the point I to the "choking" point (denoted by suffix c).

3.2. *Cases where Fully Established Pipe Flow occurs before "Choking."*—It is first assumed that the distribution of ρu in the boundary layer is given by

$$1 - \frac{\rho u}{\mathscr{R} U} = \frac{2 \cdot 5}{\zeta} \log_e \frac{\delta}{y} . \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (24)$$

It will be seen that this assumed distribution reduces to the accepted distribution for u in the boundary layer of a cylindrical pipe in incompressible flow[4] if we put $\rho = \mathscr{R}$. It follows from the definition of δ^* that

$$\delta^* = \frac{2 \cdot 5}{\zeta} \left[\delta - \frac{\delta^2}{4} \right] .$$

At the point at which fully established pipe flow begins (denoted by suffix p) we have

$$\delta_p = 1,$$

and hence

$$\theta_p = \delta^*_p / 1 \cdot 4 = 1 \cdot 339 / \zeta_p . \quad .. \quad .. \quad .. \quad .. \quad .. \quad (25)$$

It follows from equation (15) that ζ_p satisfies the equation

$$e^{B\zeta_p} - 1 = \frac{5 \cdot 456 \, R_e}{\zeta_p - 3 \cdot 749} \quad . \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (26)$$

This equation can be solved graphically to give ζ_p (and hence U_p) for the various values of the entry Reynolds number (R_e) in which we are interested. Where U_p is attained before the value of U for which $G(U, M_0) = 0$, we obtain pipe flow before the "choking" point is reached. Tables 2 and 3 indicate where this happened for the cases considered. It appears that when $R_e = 10^4$, pipe flow occurs before choking for values of M_0 less than $0 \cdot 6$, and also for values greater than some value in the range $1 \cdot 5 - 2 \cdot 0$. For $R_e = 10^6$ and 10^8, pipe flow occurs for values of M_0 less than $0 \cdot 7$, and also for values greater than $1 \cdot 25$.

For such cases equation (13) was solved up to $U = U_p$, as described in section 2.1 above, beyond that point the theory was developed as follows.

Reverting to dimensional units, the momentum equation for fully developed pipe flow becomes

$$\frac{d}{dx} \int_0^r \pi (r - y) \rho u^2 dy + \frac{dp}{dx} \pi r^2 + \frac{2\pi}{\zeta^2} r \mathscr{R} U^2 = 0 , \quad .. \quad .. \quad .. \quad .. \quad (27)$$

where $\mathscr{R}$, and U now denote the density and velocity along the axis of the pipe. The energy equation is assumed unchanged, viz.

$$\frac{\gamma}{\gamma - 1} \frac{p}{\mathscr{R}} + \frac{1}{2} U^2 = \text{const.} = C, \text{ say} . \quad .. \quad .. \quad .. \quad .. \quad .. \quad (28)$$

Continuity of rate of mass flow gives

$$\int_0^r 2\pi\,(r-y)\,\rho u\,dy = \text{const.} = Q \quad \ldots \quad (29)$$

Corresponding to the condition of constancy of velocity profile found in incompressible flow, it will be assumed that

$$\int_0^r 2\pi\,(r-y)\,\rho u^2\,dy = K_1.\mathscr{R}U^2\pi r^2 \quad \ldots \quad (30)$$

and

$$\int_0^r 2\pi\,(r-y)\,\rho u\,dy = K_2.\mathscr{R}U\pi r^2 = Q\,, \quad \ldots \quad (31)$$

where K_1 and K_2 are constants independent of x.

It will further be assumed that once fully established pipe flow has begun ζ remains constant.

Differentiating equation (28) we have.

$$\frac{dp}{dx} = \frac{\gamma-1}{\gamma}\left\{\mathscr{R}'C - \frac{d}{dx}\left(\frac{\mathscr{R}U^2}{2}\right)\right\}$$

and, since from equation (31)

$$\mathscr{R}U = \text{const.}\,,$$

$$\mathscr{R}' = -\frac{U'}{U}\,.\,\mathscr{R} = -\frac{U'}{U^2}\frac{Q}{K_2\pi r^2}\,.$$

Equation (27) can therefore be written

$$\frac{d}{dx}\left[K_1\frac{Q}{K_2}.U - \frac{\gamma-1}{2}\frac{Q}{K_2}.U\right] - \frac{\gamma-1}{\gamma}\frac{U'}{U^2}\frac{QC}{K_2} + \frac{Q}{K_2 r}\frac{2}{\zeta^2}U = 0$$

or

$$U'\left[K_1 - \frac{\gamma-1}{2\gamma} - \frac{\gamma-1}{\gamma}\cdot\frac{C}{U^2}\right] + \frac{2}{\zeta^2\gamma}\cdot U = 0\,. \quad \ldots \quad (32)$$

Hence, U' becomes infinite, and "choking" occurs, when U attains the value U_c given by

$$\frac{U_c^2}{C} = \frac{2(\gamma-1)}{2\gamma K_1 - (\gamma-1)} \quad \ldots \quad (33)$$

Equation (32) can be integrated directly to give the extra distance along the pipe to the "choking" point as

$$\frac{x_c - x_p}{r} = \frac{\zeta^2}{4\gamma}\left\{C\,(\gamma-1)\left[\frac{1}{U_p^2} - \frac{1}{U_c^2}\right] - (2\gamma K_1 - \gamma + 1)\log_e\frac{U_c}{U_p}\right\}. \quad \ldots \quad (34)$$

Using (33) this becomes

$$\frac{x_c - x_p}{r} = \frac{\zeta^2}{4\gamma}\left\{(\gamma-1)\frac{C}{U_p^2} - \frac{2\gamma K_1 - (\gamma-1)}{2}\right.$$

$$\left. - \frac{(2\gamma K_1 - \gamma + 1)}{2}\log_e\left[\frac{2(\gamma-1)}{2\gamma K_1 - (\gamma-1)}\cdot\frac{C}{U_p^2}\right]\right\}. \quad \ldots \quad (35)$$

To evaluate the right-hand side we have to find C/U_p^2 and K_1. It is easily seen that

$$\frac{C}{U_p^2} = \frac{1 + \frac{\gamma - 1}{2} M_0^2}{(\gamma - 1) M_0^2 U_p^2/U_0^2} \quad \ldots \quad (36)$$

To evaluate K_1 we note that until pipe flow is established

$$\int_0^r 2\pi (r - y) \rho u^2 \, dy =$$

$$- \mathscr{R}U^2 \int_0^r 2\pi (r - y) \frac{\rho u}{\mathscr{R}U} \left(1 - \frac{u}{U} \right) dy - \mathscr{R}U^2 \int_0^r 2\pi (r - y) \left(1 - \frac{\rho u}{\mathscr{R}U} \right) dy$$

$$+ \mathscr{R}U^2 \int_0^r 2\pi (r - y) \, dy$$

$$= \mathscr{R}U^2 \pi r^2 \left[1 - \frac{2\theta}{r} - \frac{2\delta^*}{r} \right]$$

$$= \mathscr{R}U^2 \pi r^2 \left[1 - \frac{2\theta (H + 1)}{r} \right] .$$

Hence, at the beginning of pipe flow

$$K_1 = \frac{\int_0^r 2\pi (r - y) \rho u^2 dy}{\mathscr{R}U^2 \pi r^2} = 1 - \frac{2\theta_p (H + 1)}{r} \quad \ldots \quad (37)$$

From equation (25), however,

$$\frac{\theta_p}{r} = 1 \cdot 339/\zeta_p,$$

$$\text{hence } K_1 = 1 - 6 \cdot 427/\zeta_p. \quad \ldots \quad (38)$$

By means of equations (35) (36) and (38) the extra distance beyond the beginning of pipe flow to the " choking " point can be evaluated.

4. *Results.*—In Fig. 1 the calculated distance from the entry to the " choking " point (x_c/r) for subsonic entry conditions is plotted against entry Mach number $(0 \cdot 6 - 1 \cdot 0)$ for the three entry Reynolds numbers considered (10^4, 10^6 and 10^8). The results are also detailed in Table 1A. They are shown again in Fig. 3 plotted against log R_e for entry Mach numbers of $0 \cdot 6$, $0 \cdot 7$, $0 \cdot 8$, and $0 \cdot 9$; the curve for an entry Mach number of $0 \cdot 8$ was obtained by interpolation from Fig. 1. In Figs. 5, 6 and 7 the calculated velocity, pressure, density and frictional coefficient (c_f) distributions are shown for each case considered. The corresponding results for supersonic entry conditions are shown in Figs. 2, 4, 8, 9 and 10 and Table 1B. The effect of the development of fully established pipe flow prior to " choking " for entry Mach numbers greater than about $1 \cdot 5$ is clearly shown in Fig. 4.

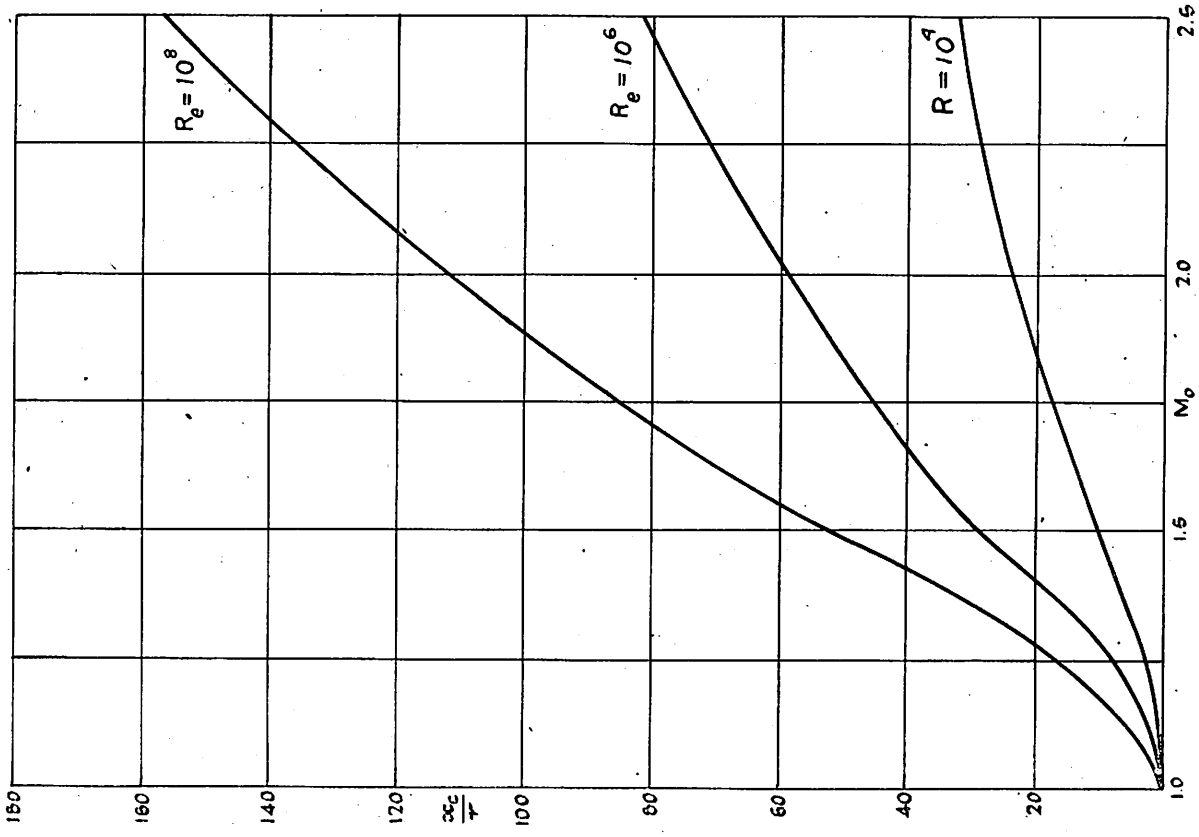
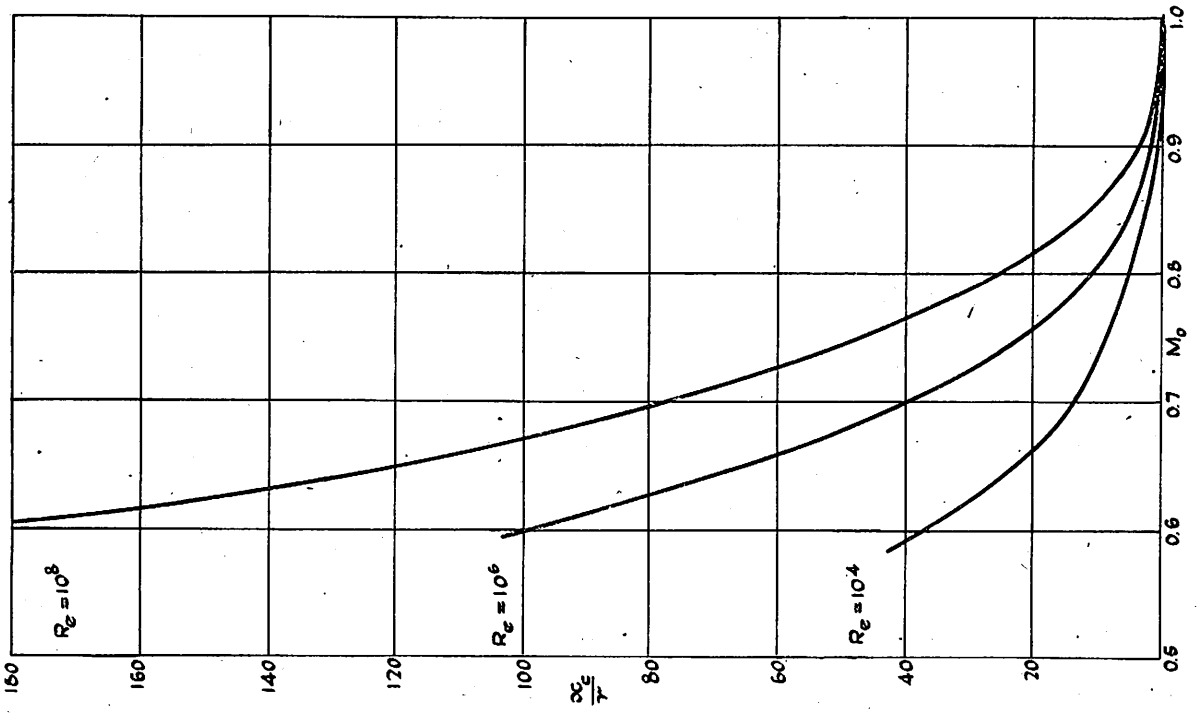


FIG. 2. Variation of "choking" length with entry Mach number for various entry Reynolds numbers (supersonic entry conditions).



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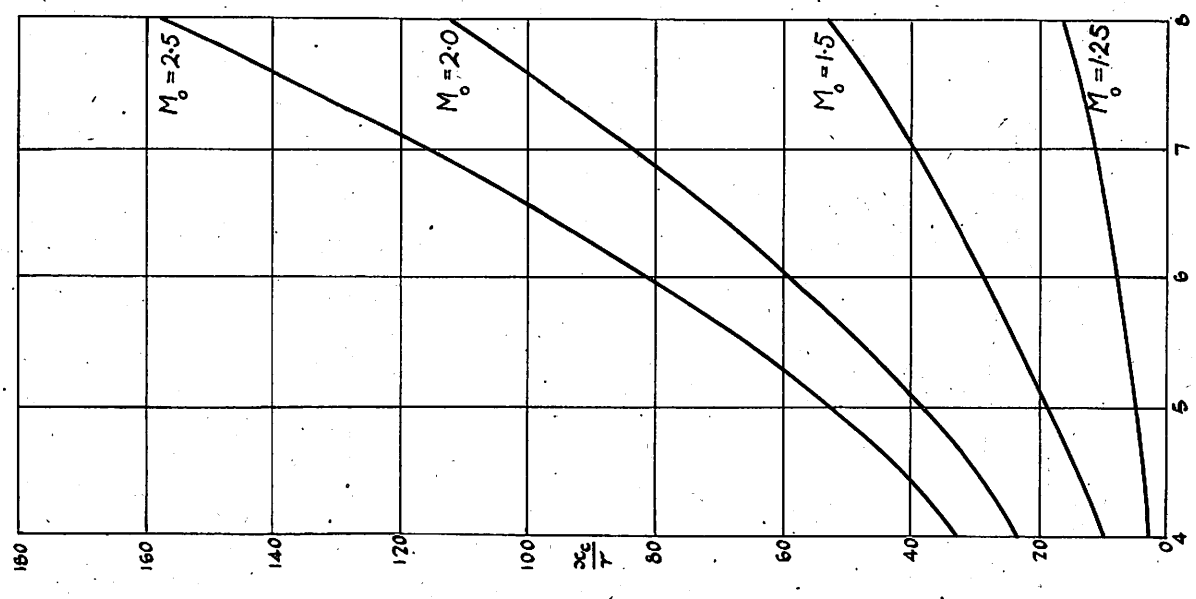

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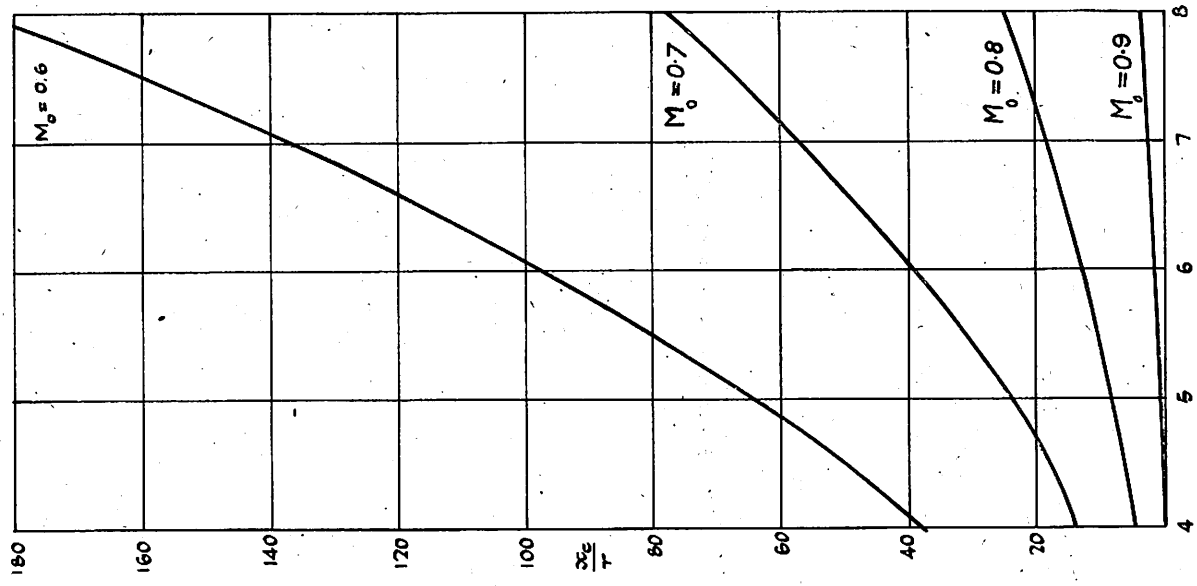

FIG. 3. Variation of "choking" length with entry Reynolds number for various entry Mach numbers (subsonic entry conditions).

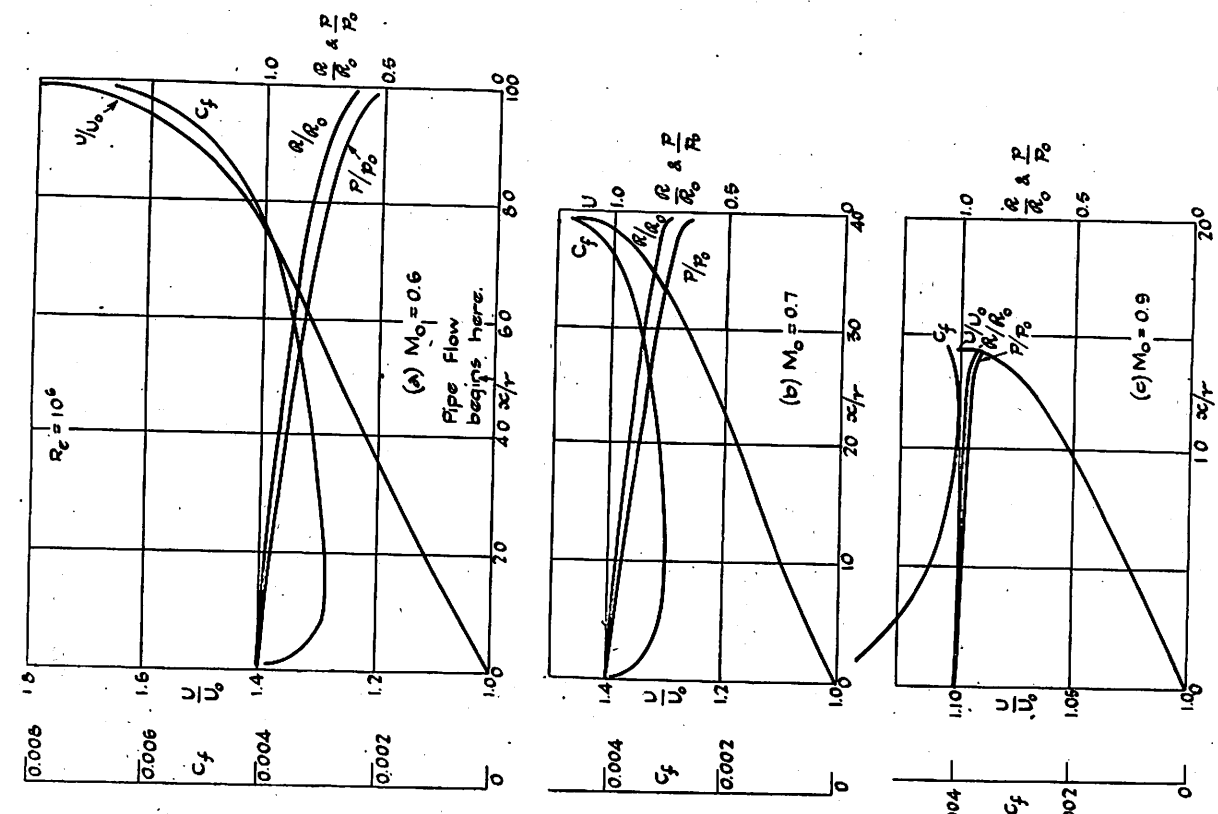

FIG. 6. Distribution along pipe of U/U_0, ρ/ρ_0, $\mathscr{R}/\mathscr{R}_0$, & c_f for $R_e = 10^6$ (*a*) $M_0 = 0{\cdot}6$, (*b*) $M_0 = 0{\cdot}7$ (*c*) $M_0 = 0{\cdot}9$.

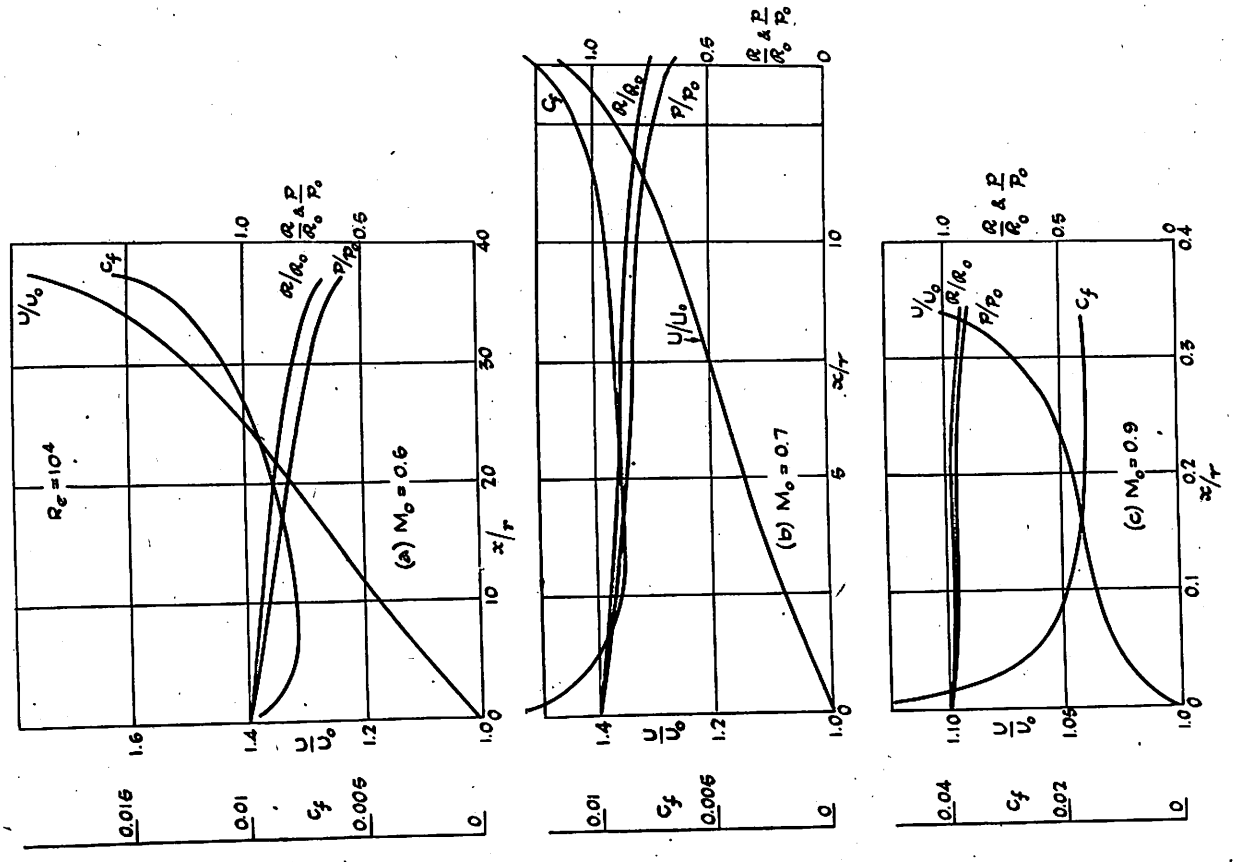

FIG. 5. Distribution along pipe of U/U_0, ρ/ρ_0, $\mathscr{R}/\mathscr{R}_0$, & c_f for $R_e = 10^4$ (*a*) $M_0 = 0{\cdot}6$ (*b*) $M_0 = 0{\cdot}7$ (*c*) $M_0 = 0{\cdot}9$.

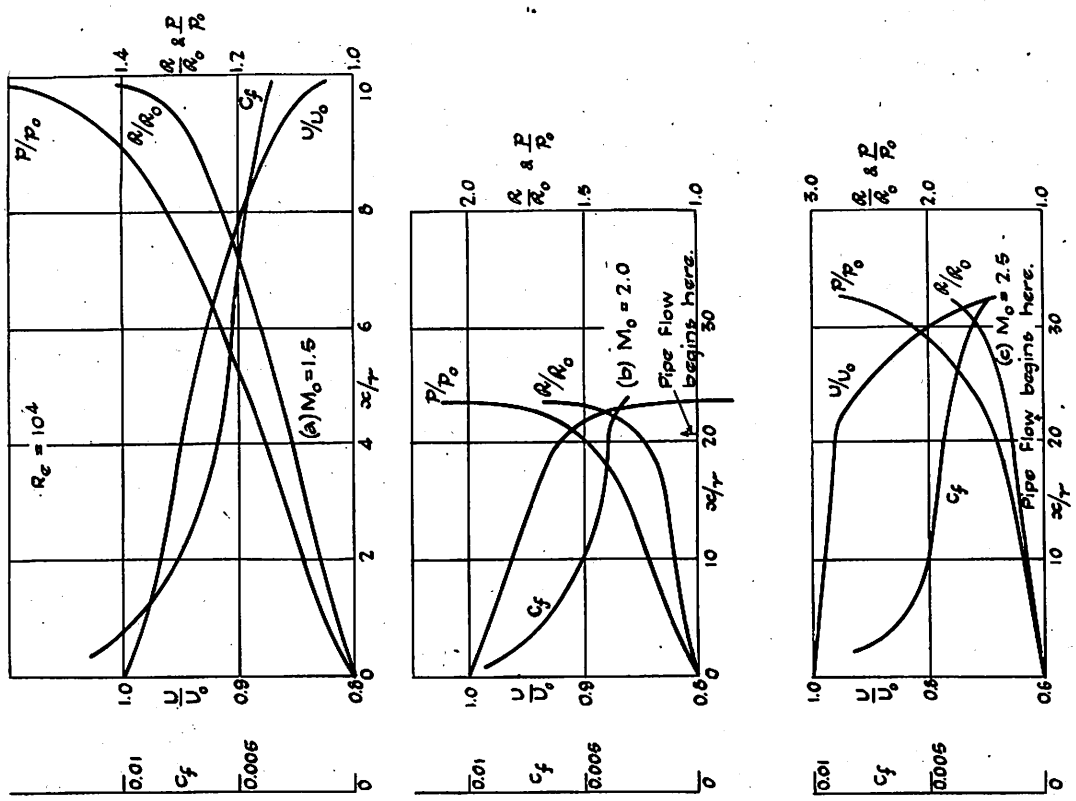

FIG. 8. Distribution along pipe of U/U_0, p/p_0, ρ/ρ_0 & c_f for $R_e = 10^4$, (a) $M_0 = 1{\cdot}5$, (b) $M_0 = 2{\cdot}0$, (c) $M_0 = 2{\cdot}5$.

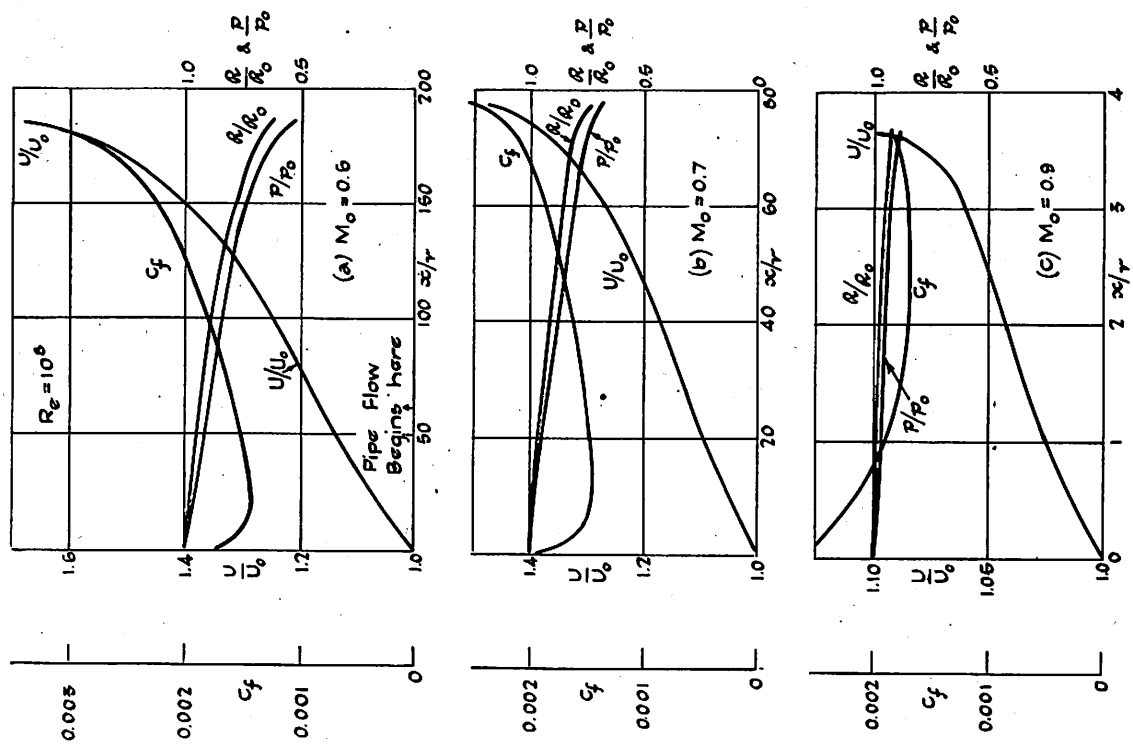

FIG. 7. Distribution along pipe of U/U_0, p/p_0, ρ/ρ_0 & c_f for $R_e = 10^6$ (a) $M_0 = 0{\cdot}6$, (b) $M_0 = 0{\cdot}7$, (c) $M_0 = 0{\cdot}9$.

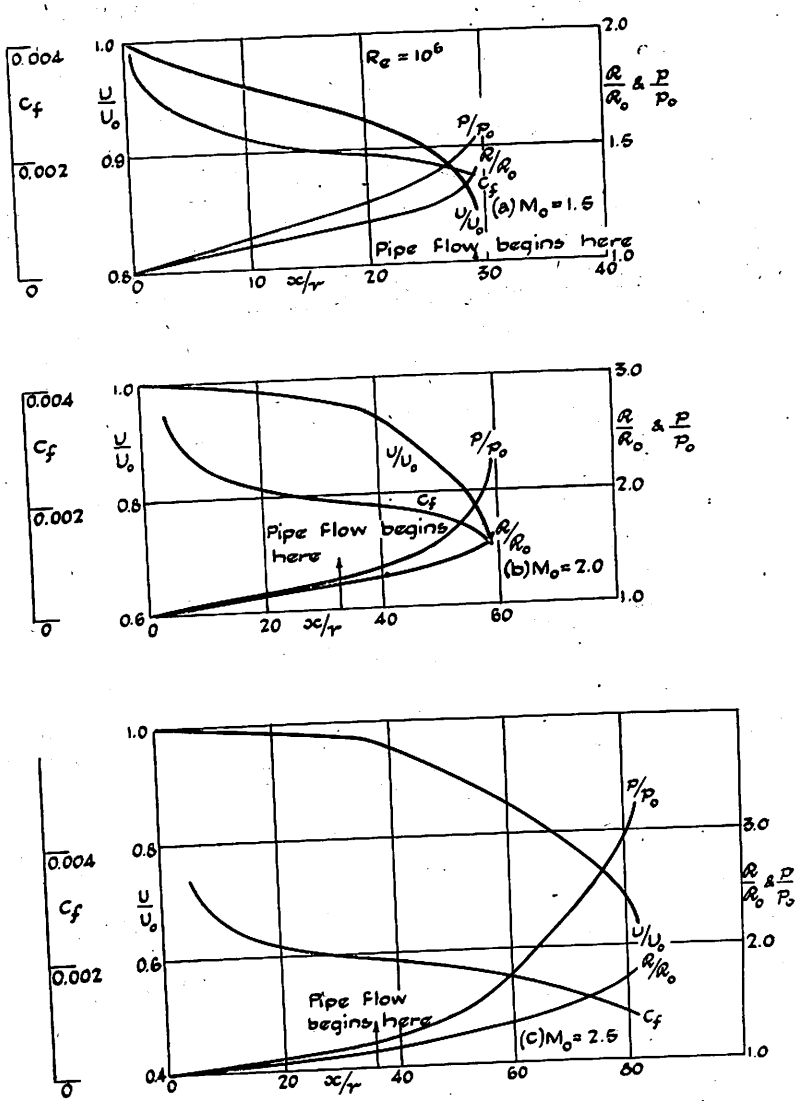

FIG. 9. Distribution along pipe of U/U_0, p/p_0, $\mathscr{R}/\mathscr{R}_0$ & c_f for $R_e = 10^6$ (*a*) $M_0 = 1{\cdot}5$, (*b*) $M_0 = 2{\cdot}0$, (*c*) $M_0 = 2{\cdot}5$.

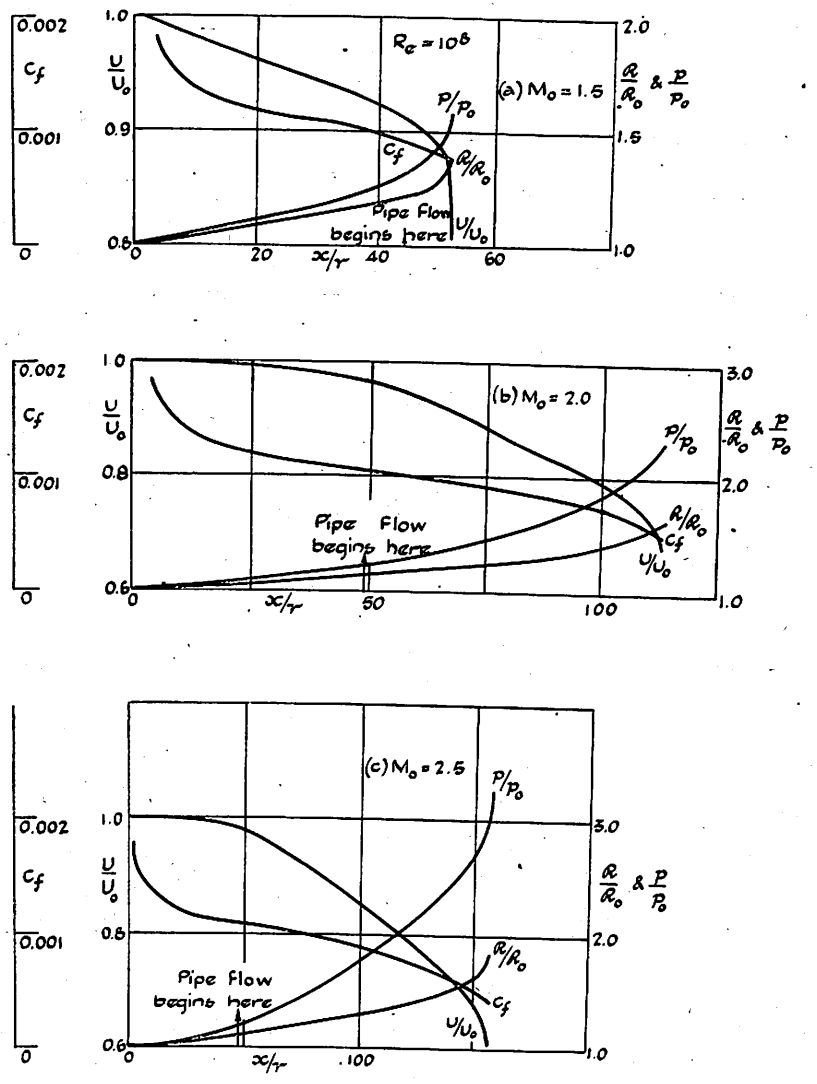

FIG. 10. Distribution along pipe of U/U_0, p/p_0, $\mathscr{R}/\mathscr{R}_0$, & c_f for $R_e = 10^8$ (*a*) $M_0 = 1{\cdot}5$, (*b*) $M_0 = 2{\cdot}0$, (*c*) $M_0 = 2{\cdot}5$.

5. *Discussion.*—Consider first the results for subsonic entry conditions. It will be seen from Figs. 1 and 3 that the "choking" length decreases with an increase of entry Mach number and with a decrease of entry Reynolds number. Consider a pipe of given dimensions and with a given entry stagnation pressure, it being assumed that the flow is varied by varying the pressure at the exit. An increase in rate of mass flow in the pipe is accompanied by an increase in entry Mach number and entry Reynolds number. The increase in entry Mach number would tend to decrease the distance to "choke," whilst the increase in entry Reynolds number would tend to increase that distance. It is easy to show from these results, however, that the effect of the increase in entry Mach number completely swamps the effect of the corresponding increase in entry Reynolds number, so that the "choking" distance decreases with increase in rate of mass flow. For the given pipe, therefore, the rate of mass flow could be increased by decreasing the pressure at the exit until the M_0 is attained for which "choking" just occurs at the end of the pipe. The question now arises as to whether, by decreasing the pressure still further at the exit, the mass flow rate M_0 can be still further increased. For this to happen "choking" must then occur inside the pipe. The breakdown of the smooth development of the flow at "choking" can occur in one of two ways, either by means of a compression shock, or else large lateral velocities develop and the streamlines cease to be parallel to the pipe axis. There are a number of reasons for ruling out the possibility of either of these occurring inside the pipe. Across a compression shock, for instance, the pressure rises; if such a shock occurs inside the pipe we are then faced with the conclusion that a reduction in pressure at the exit causes a rise in pressure there. The development of large lateral velocities would arise from the necessity of the flow which is supersonic to expand to the reduced exit pressure, this is incompatible with the presence of the pipe walls and can only occur at the exit. In any case, it is inconceivable that a change in exit pressure, which can only transmit its effect upstream at sonic velocity, can penetrate to the subsonic region at the entry and change the flow there. It can therefore be inferred that further reduction in exit pressure will have no effect on the flow inside the pipe; it can only affect what happens immediately after the flow has left the pipe. This inference is borne out by experiment; it appears that the reduction in exit pressure causes the flow to spread laterally at the exit at supersonic speeds more or less according to the Prandtl-Meyer theory (*see* Frössel[5], Stodola[6], and Aerodynamic Theory, Vol. III[4]). Generally the expansion is too great, and the pressure is reduced below the applied exit pressure, a series of compression and expansion waves follow in the jet until the pressure steadies down to the applied pressure. It follows from this that the mass flow rate reaches a maximum when the entry Mach number is that for which "choking" is just about to occur at the end of the pipe, further decrease in exit pressure has no effect on the flow inside the pipe.

Consider now the results for supersonic entry conditions (Figs. 2 and 4). Here it will be seen that the "choking" distance increases both with increase in entry Mach number and increase in entry Reynolds number. An increase in rate of mass flow now occurs with a decrease in entry Mach number, and a corresponding decrease in entry Reynolds number, and hence with a decrease in "choking" distance. A change in entry flow conditions can be brought about only by a change in the shape of the Laval nozzle which must precede the pipe to make the entry flow supersonic; increase in the ratio of the minimum cross-sectional area of the nozzle to the cross-sectional area of the pipe will reduce the entry Mach number, whilst a decrease in that ratio will increase the entry Mach number. The mass flow rate will therefore increase as the minimum cross-sectional area of the nozzle is increased, until the entry Mach number is attained for which "choking" just occurs at the pipe exit. With further increase in the nozzle cross-sectional area it appears that a compression shock develops in the pipe appearing first at the exit and then moving up the pipe towards the entry, with the mass flow rate continuing to increase. Therefore, for supersonic entry conditions the discontinuity in the flow takes the form of a compression shock. After the shock the flow becomes subsonic, and it is to be presumed that the maximum rate of mass flow is attained when "choking" for the subsonic régime occurs at the exit of the pipe. With a given nozzle shape, varying the exit pressure cannot change the mass flow rate but, as the exit pressure is increased above that at which a compression shock is about to occur there, the shock appears

and moves up the pipe towards the entry (*see* Frössel[5]). The theory of this report and the resulting curves of Figs. 2 and 4 give therefore the maximum distance for shockless supersonic flow for given entry conditions, corresponding to the case where the exit pressure is just low enough for the shock to be about to occur there. This distance has been referred to as the "choking" distance, although it does not correspond, as with the subsonic flow case, to the condition for maximum rate of mass flow, but greater rates of mass flow can only be obtained at the cost of the losses involved in a compression shock in the pipe.

6. *Comparison with Hawthorne's Results.*—Hawthorne's calculations[1] led him to the conclusion that the "choking" distance multiplied by the local frictional coefficient (assumed constant along the pipe) is a unique function of the entry Mach number. For each of these calculations, therefore, the local frictional coefficient distribution ($f = 2\tau/\rho U^2$) as averaged along the pipe length up to the "choking" point, and its product with the distance x_c is plotted against entry Mach number in Figs. 11a and b. The corresponding curves derived by Hawthorne are shown for comparison in these Figures. It will be seen that the results support to a large extent Hawthorne's conclusion that the product $\bar{f}x_c/r$ is a unique function of the entry Mach number, and further, the function derived by Hawthorne's simple analysis is fairly close for both sub- and supersonic entry conditions to that derived from the present calculations. For subsonic entry conditions Hawthorne's curve provides values of $\bar{f}x_c$ about 10 per cent. greater than is given by these calculations. For supersonic entry conditions Hawthorne's curve gives a value of $\bar{f}x_c$ that is slightly less than is given by these calculations up to an entry Mach number of about 1·5, for greater Mach numbers his value of $\bar{f}x_c$ becomes progressively too large, being about 15 per cent. too great at $M_0 = 2{\cdot}5$. These differences can be explained as follows. Until pipe flow occurs the development of the boundary layer, which Hawthorne neglects, will tend to speed up the flow in the centre of the pipe, and hence for subsonic entry conditions it tends to shorten the "choking" distance, but for supersonic entry conditions it tends to lengthen that distance. The effect of fully developed pipe flow can be understood by referring to equation (32). The essential equation of Hawthorne's analysis can be derived from equation (32) by increasing the value of K_1 to 1·0. It will be seen that for supersonic entry flow, where U' is negative, an increase in K_1, will reduce the magnitude of U' and hence lead to later "choking"; for subsonic entry flow an increase of K_1 will increase the magnitude of U' and hence lead to earlier "choking". For supersonic entry conditions pipe flow becomes fully established for entry Mach numbers above about 1·5, and its effect in hastening "choking" as compared with Hawthorne's simple theory is readily seen in Fig. 11b. For subsonic entry conditions pipe flow only becomes established for entry Mach numbers less than about 0·6; it is to be supposed that, if the calculations had been taken below that Mach number, the "choking" distance would

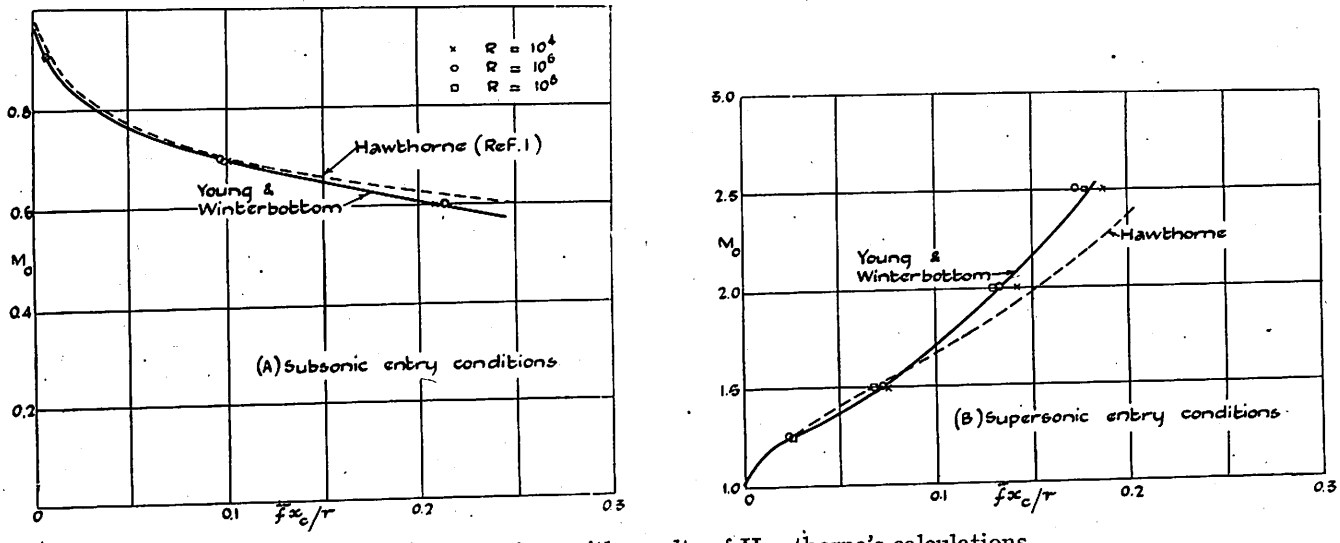

FIG. 11. Comparison with results of Hawthorne's calculations.

have steadily increased as compared with Hawthorne's estimate. It will be appreciated that, to derive a satisfactory estimate of the "choking" distance from Hawthorne's curves, the mean local frictional coefficient must be estimated with close accuracy.

7. *Comparison with Frössel's Results.*—The only experimental results available for comparison with these theoretical results are those due to Frössel described in Ref. 5. Frössel's results are given in terms of a parameter denoted by ψ/ψ_{crit}, where ψ is proportional to the actual rate of mass flow through the pipe, and ψ_{crit} is proportional to the theoretical maximum discharge rate obtained by assuming no friction in the pipe and the optimum ratio of the pressures at the ends

$$= \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} = 0{\cdot}527.$$

It can easily be shown that

$$\frac{\psi}{\psi_{crit}} = \frac{M_0}{0{\cdot}5788\left[1+\frac{(\gamma-1)}{2}M_0{}^2\right]^{(\gamma+1)/2(\gamma-1)}} \qquad (39)$$

The bulk of Frössel's experiments with subsonic entry conditions and all those with supersonic entry conditions were made with pipes of diameter 20 mm., the remainder of his experiments with subsonic entry conditions were made with pipes of diameter 10 mm., 25mm. and 30mm. His initial stagnation pressure (p_k) was in every case atmospheric. From his results he derived a family of curves, which he assumed to be universal, showing p/p_k as a function of x/r, there being a single curve for each value of ψ/ψ_{crit}; "choking" was indicated by dp/dx becoming infinite. A second family of curves was deduced from the first showing p/p_k as a function of ψ/ψ_{crit}, the parameter of the curves being the length of the pipe to the "choking" point. However, the variation of entry Reynolds number introduced in these tests by the various pipe diameters tested is relatively small; the difference between the entry Reynolds number with a pipe of 10 mm. diameter and that with a pipe of 30 mm. diameter at the same value of ψ/ψ_{crit} is equivalent to a change of 0·48 in log R_e. For most of Frössel's tests the entry Reynolds numbers were generally in the region of 10^5, from Fig. 4 a total change in log R_e of $\pm$ 0·24 about a mean of 5·0 is equivalent to a change in x/r of about $\pm$ 10 per cent. Bearing in mind the experimental difficulties and probable errors in tests of this kind, it is not surprising that Frössel ascribed the differences he found with different pipe diameters to experimental errors and assumed that his results were universal and applicable to all pipe diameters. If, in fact, his experiments had been made to cover a much wider range of entry Reynolds numbers, it is to be assumed from these calculations that systematic differences would have become apparent.

In comparing the results given by his final mean curves for "choking" distances with the results given by these calculations, the entry Reynolds numbers were therefore estimated on the assumption that the pipe diameter was 20 mm. The comparison is shown in Fig. 12 (a) for subsonic entry conditions and in Fig. 12 (b) for supersonic entry conditions, in each case x_c/r is shown plotted against ψ/ψ_{crit} and M_0. It will be seen that the agreement for subsonic entry conditions is extremely good, for supersonic entry conditions the agreement is fair for Mach numbers greater than about 1·7 and is poor for Mach numbers below 1·7. This poor agreement for these Mach numbers is not altogether surprising. Firstly, to obtain supersonic conditions the pipe was necessarily preceded by a nozzle and this nozzle would cause the entry flow to be somewhat different from the uniform flow assumed in these calculations, and will effectively increase the pipe length and so reduce the apparent "choking" length. It is to be expected that this reduction will be more important for entry Mach numbers where the total "choking" length is in any case small. Secondly, the smallest pipe length used in the tests was twenty-four times the radius; it follows that distances to the shock less than this distance would be shortened by the throttling effect of the friction of the extra pipe length beyond the shock acting in much the same way as an increased pressure at the exit. Hence, Frössel's results for values of $x_c/r < 24$ (or $M_0 < 1{\cdot}7$) cannot be accepted as of general validity.

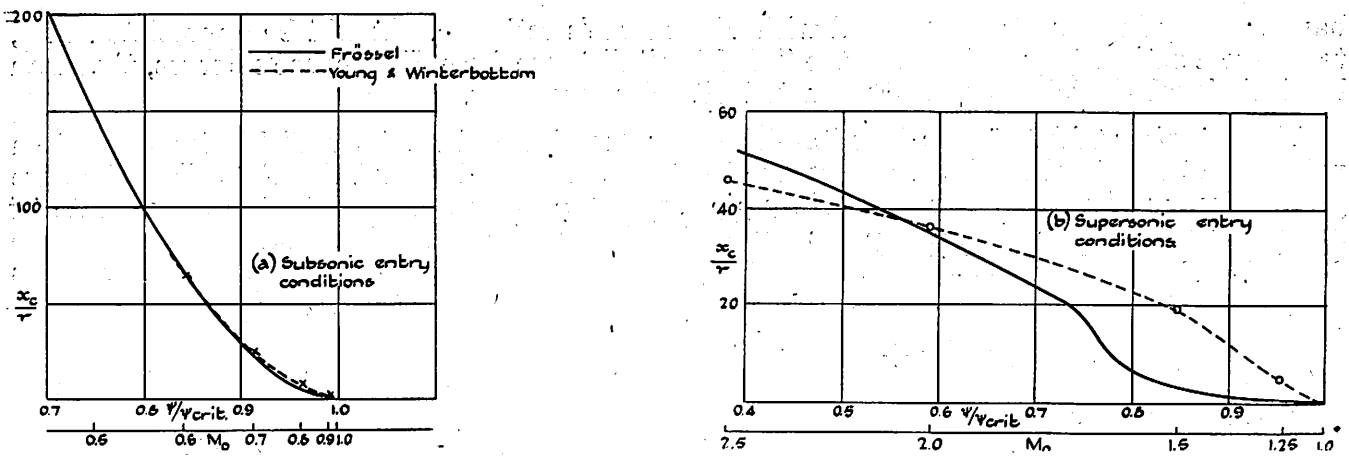

FIG. 12. Comparison of Frössel's results with theoretical results for " choking " distance with varying entry Mach number.

APPENDIX

Arithmetic Mean Square Mach Number at " Choking."—In view of the fact that according to Hawthorne's simplified theory (assuming uniform velocity and density across any pipe section) " choking " occurs at a local Mach number of unity it was thought of some interest to calculate the arithmetic mean square Mach number at " choking " given by these calculations.

Denoting this mean square Mach number by $\bar{M}_c^2$, we have

$$\bar{M}_c^2 = \int_0^r \frac{2\pi\,(r-y)}{\pi r^2}\,\frac{\rho u^2}{\gamma p}\,.\,dy \quad . \qquad (40)$$

Consider first the case where there is no pipe flow. Then

$$\int_0^r 2\pi\,(r-y)\,\rho u\,dy = \mathscr{R}U^2\pi r^2 - \mathscr{R}U^2\,(\delta^* + \theta)\,2\pi r \quad .$$

Hence
$$\bar{M}_c^2 = M_c^2\left[1 - 2\,(H+1)\,\frac{\theta}{r}\right] \quad . \qquad (41)$$

where
$$M_c^2 = \mathscr{R}U^2/\gamma p \quad .$$

Where pipe flow occurs before " choking ", we have from equations (37) and (40)

$$\bar{M}_c^2 = M_c^2 \cdot K_1 \quad . \qquad (42)$$

The calculated values of $\bar{M}_c^2$ for these cases considered are shown in Table 1. It will be seen that in every case $\bar{M}_c^2$ is greater than unity although only slightly so in most cases.

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TABLE 1

" Choking " Lengths and Conditions at " Choking " Point for the Various Cases Considered.

R_e	M_0	x_c/r	U_c/U_0	M_c	$\bar{M}_c^2$	Remarks.
(A) Subsonic entry conditions.						
10^4	0·6	37·2	1·770	1·156	1·009	No pipe flow before " choking."
10^4	0·7	13·9	1·478	1·101	1·027	No pipe flow before " choking."
10^4	0·9	0·339	1·101	1·008	1·004	No pipe flow before " choking."
10^6	0·6	98·4	1·807	1·186	1·112	Pipe flow established at $x/r = 50·5$.
10^6	0·7	39·1	1·478	1·101	1·027	No pipe flow before " choking."
10^6	0·9	1·46	1·101	1·008	1·004	No pipe flow before " choking."
10^8	0·6	185·6	1·734	1·129	1·078	Pipe flow established at $x/r = 77·5$.
10^8	0·7	77·5	1·478	1·101	1·027	No pipe flow before " choking."
10^8	0·9	3·66	1·101	1·008	1·004	No pipe flow before " choking."
(B) Supersonic entry conditions						
10^4	1·25	2·38	0·873	1·053	1·020	No pipe flow before " choking."
10^4	1·50	10·28	0·822	1·151	1·019	No pipe flow before " choking."
10^4	2·0	23·7	0·769	1·334	1·224	Pipe flow established at $x/r = 20·3$.
10^4	2·5	32·4	0·688	1·334	1·224	Pipe flow established at $x/r = 22·0$.
10^6	1·25	7·59	0·873	1·053	1·020	No pipe flow before " choking."
10^6	1·50	29·2	0·842	1·186	1·112	Pipe flow established at $x/r = 29·1$.
10^6	2·0	59·0	0·704	1·186	1·112	Pipe flow established at $x/r = 34·6$.
10^6	2·5	81·5	0·630	1·186	1·112	Pipe flow established at $x/r = 36·0$.
10^8	1·25	16·34	0·873	1·053	1·020	No pipe flow before " choking."
10^8	1·50	52·6	0·809	1·129	1·078	Pipe flow established at $x/r = 44·8$.
10^8	2·0	112·6	0·677	1·129	1·078	Pipe flow established at $x/r = 49·0$.
10^8	2·5	157·6	0·606	1·129	1·078	Pipe flow established at $x/r = 49·4$.

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